33B Final

Vedant Sahu

TOTAL POINTS

94 / 100

QUESTION 1

1 homogenous eon 7 / 7

+ **2 pts** Homogeneous
+ **1 pts** Substitution
✓ + **3 pts** Single-Variable Integrating Factor
✓ + **2 pts** Making Exact
✓ + **2 pts** Solving
+ **0 pts** No Points

QUESTION 2

2 separable eqn 4 / 5

- **0 pts** Correct
- **1 pts** minor mistake
✓ - **1 pts** need more simplification
- **5 pts** no work
- **3 pts** know it's separable equation, fail to do the partial fraction decomposition
- **2 pts** right hand side is polynomials of x, integration can be calculated directly.
- **3 pts** idea is correct, need calculation

QUESTION 3

forcing term 10 pts

3.1 polynomial 2 / 2

✓ - **0 pts** Correct
- **0.5 pts** b=-1
- **0.5 pts** c=-1
- **0.5 pts** a=2
- **2 pts** wrong

3.2 sin 4 / 4

✓ - **0 pts** Correct
- **1 pts** missing cos in the Setup/t for cos in Setup/wrong second setup
- **1 pts** computational mistake
- **2 pts** missing 2 in the differential
- **4 pts** wrong/no answer
- **2 pts** missing second step
- **2 pts** computational mistake
- **0.5 pts** missing t in answer
- **0.5 pts** missing - in the answer
- **1 pts** didn't finish
- **1 pts** missing 1 step

3.3 general solution 4 / 4

✓ - **0 pts** Correct
- **1 pts** no/wrong characteristic polynomial
- **1 pts** no/wrong roots
- **1 pts** no/wrong homogeneous solution
- **1 pts** wrong final answer
- **4 pts** wrong/no answer
- **0.5 pts** - missing for polynomial
- **0.5 pts** missing t for cos/wrong answer for trig part

QUESTION 4

4 system 10 / 10

✓ - **0 pts** Correct
- **1 pts** Incorrectly identified the eigenvalues or their algebraic multiplicity.
- **2 pts** Incorrectly found the eigenvectors.
- **2 pts** Incorrectly found generalized eigenvectors.
- **2 pts** Incorrect coefficients or powers of t or (A-L I) in general solution
- **1 pts** Wrong vectors in general solution.
- **2 pts** Failed to solve IVP.
- **1 pts** Arithmatic error
- **1 pts** Got an unsolvable system when solving IVP.

QUESTION 5

2nd linear differential equation 8 pts

5.1 verify 4 / 4

✓ **- 0 pts** Correct

- 2 pts incorrect calculation

- 2 pts not finished

- 4 pts no work

- 3 pts some work

5.2 find general solution 4 / 4

✓ **- 0 pts** Correct

- 4 pts Incorrect calculation of homogeneous. For second order linear differential equation, use y_g=c1*y_h1+c2*y_h2+y_p. y1,y2,y3 can be decomposed in that way, hence we can get y_h1=y1-y2, y_h2 = y2-y3.

- 2 pts incorrect calculation of y_h1, y_h2, but idea is correct

- 3 pts incorrect calculation of y_h1, y_h2

- 3 pts some work

- 1 pts y_g=c1*y_h1+c2*y_h2+y_p

- 1 pts no calculation detail

QUESTION 6

linear system 9 pts

6.1 find general solution 5 / 5

✓ **- 0 pts** Correct

- 3 pts eigenvector: solve for (A-\lambda*I)v = 0.

- 2 pts some calculation error\no finished

- 1 pts final answer incorrect

- 5 pts no work

- 3 pts calculation error, incorrect eigenvalue, eigenvector, idea is correct

6.2 spiral? 1 / 1

✓ **- 0 pts** Correct

- 1 pts incorrect

6.3 sink\source? 1 / 1

✓ **- 0 pts** Correct

- 1 pts incorrect

6.4 direction? 2 / 2

✓ **- 0 pts** Correct

- 2 pts wrong

- 1 pts Somework

QUESTION 7

7 8 / 9

- 0 pts Correct

✓ **- 1 pts** one entry wrong

- 2 pts two entries wrong

- 3 pts three entries wrong

- 4 pts 4 entries wrong

- 5 pts 5 entries wrong

- 7 pts all entries wrong

- 1 pts incorrect initial value

- 2 pts initial value missing

- 9 pts wrong/ no answer

- 2 pts not taking concentration

QUESTION 8

8 pts

8.1 3 / 3

✓ **+ 1 pts** Correct Roots

✓ **+ 2 pts** Phase Line

+ 0 pts No Points

8.2 3 / 3

✓ **+ 1 pts** Curves

✓ **+ 1 pts** 1 Stability

✓ **+ 1 pts** 3 Stability

+ 0 pts No Points

8.3 2 / 2

✓ **+ 1 pts** Correct

✓ **+ 1 pts** Justification

+ 0 pts No Points

QUESTION 9

9 pts

9.1 4 / 4

✓ **- 0 pts** Correct

- 4 pts Didn't know to solve x^2-3x+2=0.

- **2 pts** Got the wrong roots.
- **2 pts** didn't write solutions

9.2 3 / 3

✓ **- 0 pts Correct**

- **1 pts** Didn't mention uniqueness theorem.
- **2 pts** Didn't say that uniqueness means solution cannot cross the solutions x=1 and x=2.

9.3 2 / 2

✓ **- 0 pts Correct**

- **1 pts** Wrong answer
- **1 pts** Inadaquate justification.

QUESTION 10

9 pts

10.1 3 / 3

✓ **- 0 pts Correct**

- **1 pts** Didn't use the definition of exact.
- **2 pts** Incorrectly solved for b and m.
- **1 pts** Incorrectly solved for one of b or m.

10.2 6 / 6

✓ **- 0 pts Correct**

- **1 pts** Minor Calculation error.
- **2 pts** Found antiderivative, but not solution (need to set F(x,y)=C).
- **2 pts** Did not use the correct algorithm to solve.
- **6 pts** Wrong/Blank
- **1 pts** Incorrectly solved for g'(y) or h'(x).

QUESTION 11

11 pts

11.1 3 / 3

✓ **+ 1.5 pts Correct for A**

✓ **+ 1.5 pts Correct for B**

+ **0 pts** No Points

11.2 2 / 4

✓ **+ 2 pts Eigenvector**

+ **2 pts** Sketch
+ **0 pts** No Points

(0,1) is not a half-line solution

11.3 2 / 4

✓ **+ 2 pts Two Eigenvectors**

+ **2 pts** Star Behavior
+ **0 pts** No points

Every vector is a half-line solution

QUESTION 12

5 pts

12.1 2 / 2

✓ **- 0 pts Correct**

- **0.5 pts** did not solve for v' (correctly)
- **2 pts** no answer/ wrong answer
- **1 pts** wrong substitution

12.2 3 / 3

✓ **- 0 pts Correct**

- **3 pts** wrong/no answer
- **2 pts** for trying
- **1 pts** missing F(t, y, ... , y_{n-1}) in answer/missing ' in the answer
- **1.5 pts** only using one variable
- **1 pts** missing equations in answer
- **1 pts** using y^{i} in equations
- **1 pts** not adding additional equations

74

FINAL

12/10/2018

Math33B
Nadja Hempel
nadja@math.ucla.edu

Name: VEDANT SAHU

UID: 604 997 878

section: 2B

Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

$P = x^2 + y^2 \qquad Q = -2xy$

$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = -2y$

$$h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{-2xy}(2y + 2y) = -\frac{2}{x}$$

$$\mu(x) = e^{\int h(x)\,dx} = e^{\int -2/x\,dx}$$

$$= e^{-2\ln|x|} = 1/x^2$$

$$\frac{(x^2+y^2)}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0$$

$$\left(1 + y^2/x^2\right)dx - 2y/x\,dy = 0$$

This equation is exact

$$F(x,y) = \int Q(x,y)\,dy + \phi(x)$$

$$= -y^2/x + \phi(x)$$

$$\frac{\partial F}{\partial x} = \cancel{-}\frac{\cancel{2}y^2}{x^2} + \phi'(x) = \cancel{1}\ \cancel{P} = \cancel{y^2}\ \mu(x)\,P$$

$$\Rightarrow \phi'(x) = \cancel{1+}\ \cancel{\frac{3y^2}{x^2}}\ 1 + y^2/x^2 - y^2/x^2 = 1$$

So, $\phi(x) = x$

Therefore, $F(x,y) = -y^2/x + x$

$F(x,y) = C$ is the required solution.

$-y^2/x + x = C$

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$\int \frac{dy}{y(y+1)} = \int (x+2)(x+3)\,dx$$

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$A(y+1) + By = 1$$

$$\Rightarrow A = 1, \quad B = -1$$

$$\int \left(\frac{1}{y} - \frac{1}{y+1} \right) dy = \int (x^2 + 5x + 6)\,dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = x^3/3 + 5x^2/2 + 6x + C_1$$

$y \neq 0$
$y \neq -1$

$$\Rightarrow \left| \frac{y}{y+1} \right| = e^{x^3/3 + 5x^2/2 + 6x + C_1}$$

$$\Rightarrow \frac{y}{y+1} = A_1 e^{(x^3/3 + 5x^2/2 + 6x)} \qquad A_1 \in \mathbb{R}$$

$\cancel{\Rightarrow} \ \cancel{\frac{y+1}{y}}$ This is the impelicitely defined solution.

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

Let $y(t) = a_0 + a_1 t + a_2 t^2$

$y'(t) = a_1 + 2a_2 t$ $\quad y''(t) = 2a_2$

$y'' + 4y = 8t^2 - 4t$

$\Rightarrow 2a_2 + 4a_0 + 4a_1 t + 4a_2 t^2 = 8t^2 - 4t$ $\quad a_2 = 2$

Comparing both sides, we get $\quad a_1 = -1$

$4a_2 = 8,\ 4a_1 = -4,\ 4a_0 = \cancel{-16} - 2a_2$ $\quad a_0 = -4/4$

Therefore, $y_P(t) = \cancel{8t^2 - 4t - 16}\ 2t^2 - t - \cancel{4}\ 1$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

Let $y(t) = a\sin(2t) + b\cos(2t)$

$y'(t) = 2a\cos(2t) - 2b\sin(2t)$

$y''(t) = -4a\sin(2t) - 4b\cos(2t)$

$y'' + 4y = 4\sin(2t)$

$\Rightarrow -4a\sin(2t) - 4b\cancel{s}\cos(2t) + 4a\sin(2t)$

$+ 4b\sin(2t) = 0 \neq 4\sin(2t)$

Let $y(t) = at\sin(2t) + bt\cos(2t)$

$y'(t) = a\sin(2t) + b\cos(2t) + 2at\ \cancel{\sin(2t)}\ \cos(2t)$

$- 2bt\ \cancel{\cos(2t)}\ \sin(2t)$

$y''(t) = \cancel{4a\sin(2t)}\ 4a\cos(2t) - 4b\sin(2t)$

$- 4at\sin(2t) - 4bt\cos(2t)$

Now, $y'' + 4y = 4\sin(2t)$

$\Rightarrow 4a\cos(2t) - 4b\sin(2t) - 4at\sin(2t)$

$- 4bt\cos(2t) + 4at\sin(2t) + 4bt\sin(2t)$

$= 4\sin(2t)$

$\Rightarrow 4a\cos(2t) - 4b\sin(2t) = 4\sin(2t)$

Comparing both sides, we get

$a = 0,\quad b = \cancel{1}\ -1$

Therefore, $y_P(t) = -t\cos(2t)$

$\cancel{-\cos(2t) - 2t\sin(2t)}$

$\cancel{-4\sin(2t) \neq -4t\cos(2t)}$

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

$$\begin{aligned} y_p(t) &= 2\, y_{p2}(t) - y_{p1}(t) \\ &= 2(-t\cos(2t)) - (2t^2 - t - 1) \\ &= -2t\cos(2t) - 2t^2 + t + 1 \end{aligned}$$

$$y'' + 4y = 0$$

$$\lambda^2 + \cancel{4\lambda} \; 4 = 0$$

$$\lambda = \pm 2i \qquad a = 0, \; b = 2$$

$$y_1(t) = e^{0t}\cos(2t) = \cos(2t)$$

$$y_2(t) = e^{0t}\sin(2t) = \sin(2t)$$

Therefore, the general solution is given by

$$\begin{aligned} y_g(t) &= y_p(t) + C_1 y_1(t) + C_2 y_2(t) \\ &= -2t^2 + t + 1 - 2t\cos(2t) + C_1\cos(2t) \\ &\quad + C_2\sin(2t) \end{aligned}$$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

This is an upper ~~matrix~~ triangular matrix

So, $\det(A - \lambda I)$ = Product of diagonal entries

$$\det(A - \lambda I) = (-1-\lambda)(-1-\lambda)(3-\lambda)(-1-\lambda) = 0$$

Eigenvalues : $\cancel{\lambda = -1 \text{ wi}}$

$\lambda_1 = -1$ with alg. mult. $= 3$

$\lambda_2 = 3$ with alg. mult $= 1$

$$E_3 = \ker(A - 3I) = \ker \begin{bmatrix} -4 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$\cancel{2\vec{c_1} + 4\vec{c_2} + 16\vec{c_3} = 0}$

$\vec{c_1} + 2\vec{c_2} + 8\vec{c_3} = 0$

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} \in E_3$$

$$E_{-1} = \ker(A + I) = \ker \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(\ker(A + I)) = 1$

But, we need 3 vectors

$$(A + I)^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A + I)^3 = \begin{bmatrix} 0 & 0 & 8 & -6 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need to pick $\vec{v_2}$ such that $\vec{v_2} \in \ker (A+I)^3$

but ~~$\vec{v_2} \notin \ker (A+I)^2$~~ $\vec{v_2} \notin \ker (A+I)^2$

7

$3\vec{c_3} + 4\vec{c_4} = \vec{0}$ So let $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

$$\vec{y_1}(t) = \cancel{c_1}\, e^{-t} \left(\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\vec{y_2}(t) = e^{-t} \left(\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \qquad \vec{y_3}(t) = e^{-t} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{y_4}(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix}$$

$$\vec{y}(t) = c_1 \vec{y_1}(t) + c_2 \vec{y_2}(t) + c_3 \vec{y_3}(t) + c_4 \vec{y_4}(t)$$

This is the general solution

$$\vec{y}(0) = c_1 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -6 \\ -2 \end{bmatrix}$$

$4c_1 = -2 \Rightarrow c_1 = -1/2$

$3c_1 + 8c_4 = -6 \Rightarrow c_4 = -15/16$

$3c_2 + 2c_4 = 1 \Rightarrow c_2 = 23/24$

$6c_3 + c_4 = 3 \Rightarrow c_3 = 21/32$

Therefore, the solution to the initial value problem is

$$\vec{y}(t) = -\frac{1}{2}\vec{y_1}(t) + \frac{23}{24}\vec{y_2}(t) + \frac{21}{32}\vec{y_3}(t) - \frac{15}{16}\vec{y_4}(t)$$

$\cancel{2 \cancel{8} c_3 = 3 + \frac{15}{16} = \frac{\cancel{63}\,21}{16}}$

$\cancel{8c_4 = -7.5 = -\frac{15}{2}}$

$\cancel{c_4 \quad 1 + \frac{15}{5} = \frac{23}{8}}$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t - 1) - t(e^t + 1), \qquad (t > 0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt) (Show ALL your calculations in detail for full credit)

$y_1 = e^t(2t+1) - (t+1)$

$\cancel{y_1'}$ $\cancel{y_1''}$ $y_1' = e^t(2t+1) + 2e^t - 1$

$y_1'' = e^t(2t+1) + 4e^t$

$t^2 y_1'' - (t^2+2t)y_1' + (t+2)y_1$

$= t^2(2t+1)e^t + 4t^2e^t - (t^2+2t)e^t(2t+1)$

$\quad - (t^2+2t)(2e^t-1) + (t+2)e^t(2t+1)$

$\quad - (t+1)(t+2)$

$= \cancel{t^2e^t} + 4t^2e^t - \cancel{2}t\,e^t(2t+1) - \cancel{t^2}\; 2t^2e^t - 4te^t$

$\quad \cancel{+}\, t^2 + 2t + \cancel{2te^t(2t+1)} + \cancel{9}\; 4te^{4t} + 2e^t$

$\quad \cancel{-t^2} - 3t - 2 + \cancel{t^2(2t+1)e^t} - \cancel{t^2(2t+1)e^t}$

$= 2\cancel{3}t^2e^t - 2t^2e^t - \cancel{2}te^t \cancel{-2e^t} - t - 2 + 2e^t$

$= 2(e^t - 1) - t(e^t + 1)$

Therefore, $y_1 = e^t(2t+1) - (t+1)$ is a solution.

Hence, proved.

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

For the general solution, we need a particular solution as well as the two solutions of the homogenous equation.

y_1, y_2, and y_3 are ~~ats~~ all particular solutions. We get ~~θ~~ the solution of the homogenous eqn. by ~~sub~~

$$
\begin{aligned}
y_{h1} &= \cancel{y_p =} \; y_1 - y_2 \\
&= e^t(2t+1) - (t+1) - e^t(t+1) - (t-1) \\
&= te^t - 2t
\end{aligned}
$$

$$
\begin{aligned}
y_{h2} &= y_1 - y_3 \\
&= e^t(2t+1) - (t+1) - e^t(1-t) - (2t-1) \\
&= 3te^t - 3t
\end{aligned}
$$

$y_{h1}(t)$ and $y_{h2}(t)$ are linearly ~~de~~ independent as there is no $C \in \mathbb{R}$ for which $y_{h1} = C\, y_{h2}$

Therefore, the general solution is given by

$$
\begin{aligned}
y(t) &= y_1(t) + C_1 y_{h1}(t) + C_2 y_{h2}(t) \\
&= e^t(2t+1) - (t+1) + C_1(te^t - 2t) \\
&\quad + C_2(3te^t - 3t)
\end{aligned}
$$

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

$$A - \lambda I_2 = \begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_2) = (\lambda - 1)(\lambda - 3) + 2 = \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\lambda_1 = 2+i \quad , \quad \lambda_2 = 2-i$$

$$E_{\lambda_1} = \ker(A - \lambda_1 I) = \ker \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}$$

$$\vec{v_1} = \begin{pmatrix} -2 \\ 1+i \end{pmatrix} \in E_{\lambda_1}$$

$$\vec{v_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{w_1} + i\,\vec{w_2}$$

~~So = $\vec{v_2}$ =~~

The general solution is given by

$$\vec{y}(t) = \cancel{e}\; c_1 e^{2t} \left(\cos t \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$+ \; c_2 e^{2t} \left(\sin t \begin{pmatrix} e\;\; -2 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

(2) Conclude that the equilibrium point is a spiral. (1pt)

$T^2 - 4D < 0$

We know that this lies above the parabola

$T^2 - 4D = 0$ in the Trace - Determinant plane.

Therefore, it is a spiral.

$c_1 (\cos t \, \vec{w_1} - \sin t \, \vec{w_2}) + c_2 (\sin t \, \vec{w_1} + \cos t \, \vec{w_2})$

describes an ellipse.

(3) Is it a sink or a source? (1pt)

$\lambda = \alpha + i\beta$

$T = 2\alpha = 4 > 0 \qquad \alpha > 0$

As $t \to \infty \qquad e^{2t} \to \infty$

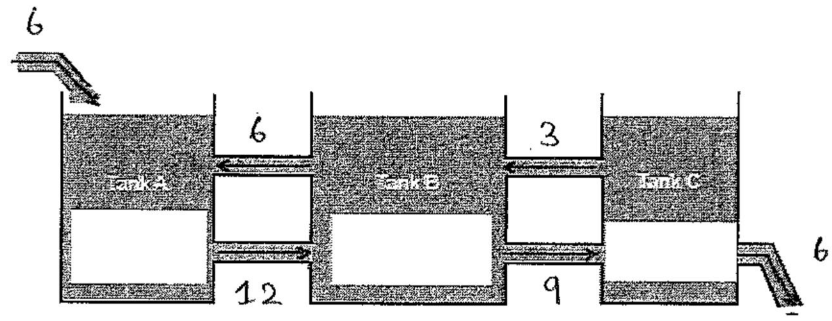
Therefore, it is a source

(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$a_{21} > 0$

Therefore, the spiral rotates counterclockwise.



Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

$$\vec{X}(0) = \begin{pmatrix} X_A(0) \\ X_B(0) \\ X_C(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$$

Volume of all 3 tanks ~~remains~~ remain the same as their initial volumes throughout the process

$$x_A' = \text{Rate in} - \text{Rate out}$$

$$= \frac{x_B}{120}\cdot 6 - \frac{x_A}{60}\cdot 12$$

$$x_B' = \text{Rate in} - \text{Rate out}$$

$$= \frac{x_A}{60}\cdot 12 + \frac{x_C}{30}\cdot 3 - \frac{x_B}{120}\cdot (6+9)$$

$$x_C' = \text{Rate in} - \text{Rate out}$$

$$= \frac{x_B}{120}\cdot 9 - \frac{x_C}{30}\cdot 3$$

$$\vec{x}' = \begin{pmatrix} x_A' \\ x_B' \\ x_C' \end{pmatrix} = \begin{pmatrix} -1/5 & 1/20 & 0 \\ 1/5 & -1/\cancel{10}8 & 1/10 \\ 0 & 3/40 & -1/10 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix} -1/5 & 1/20 & 0 \\ 1/5 & -1/8 & 1/10 \\ 0 & 3/40 & -1/10 \end{pmatrix} \vec{x}$$

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$$

Exercise 8. (8pt)
Consider the differential equation

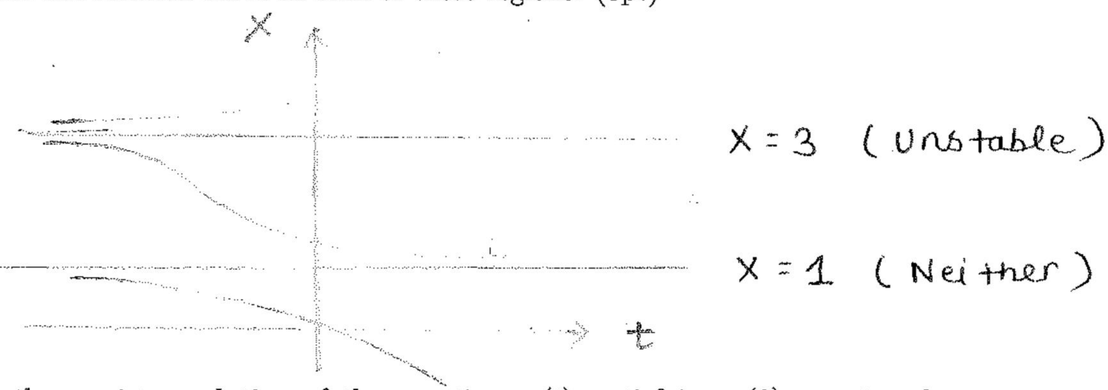
$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3)$$

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

$$\frac{dx}{dt} = e^x (x-1)(x^2 - 4x + 3) \cancel{= e^x}$$

$$\Rightarrow \quad \frac{dx}{dt} = e^x (x-1)^2 (x-3)$$

(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)



(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)

We have seen that below the equilibrium solution $x(t) = 1$, $dx/dt < 0$

$\Rightarrow$ In this region, $x(t)$ is a decreasing function

So $x(2)$ must be less than $x(0)$

$t_1 > t_2 \Rightarrow x(t_1) < x(t_2)$ for a decreasing function

But here $x(2) = 0 > x(0) = -1$

Therefore, such a solution cannot exist.

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx} = \frac{(x-2)(x-1)}{tx}$$

When $x(t) = 2$, $\frac{dx}{dt} = 0$ and when

$x(t) = 1$, $dx/dt = 0$

Therefore, $x(t) = 1$ and $x(t) = 2$ are the two constant solutions.

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

$f(t,x) = \frac{x^2 - 3x + 2}{tx}$ is continuous for all

$t \neq 0$ and $x \neq 0$. Similarly,

$\frac{\partial f}{\partial x} = \frac{(2x-3)x - x(x^2 + 3x + 2)}{tx^2}$ is continuous

for all $t \neq 0$ and $x \neq 0$. Therefore, existence and uniqueness theorem applies. (continued at the end)

(3) Can you apply the existence theorem to the initial value problem $y(0) = 5$? (1pt) Justify your answer. (1pt)

$f(t,x) = \frac{x^3 - 3x + 2}{tx}$

There is no rectangle R containing $t = 0$ in which $f(t,x)$ is continuous. Therefore, the existence theorem cannot be applied to the initial value problem $x(0) = 5$.

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$$\frac{\partial P}{\partial y} = 4xy \qquad \frac{\partial Q}{\partial x} = m b x^{m-1} y$$

~~Thi~~ The equation will be exact on the rectangle iff $\partial P/\partial y = \partial Q/\partial x$

$\Rightarrow 4xy = mbx^{m-1}y \qquad m-1 = 1 \Rightarrow m = 2$

$mb = 4 \Rightarrow b = 4/m = 2$

Therefore, $b = 2$ and $m = 2$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$$2(x + xy^2) + 2(x^2 y + y^2)\frac{dy}{dx} = 0$$

$$F(x,y) = \int P(x,y)\,dx + \phi(y)$$

$$= \int (2x + 2xy^2)\,dx + \cancel{\phi(x)}\ \phi(y)$$

$$= x^2 + x^2 y^2 + \cancel{\phi(x)}\ \phi(y)$$

Now, $\dfrac{\partial F}{\partial y} = Q(x,y)$

$\Rightarrow 2x^2 y \cancel{- 2x^2} + \phi'(y) = 2x^2 y + 2y^2$

$\Rightarrow \phi'(y) = 2y^2$

$\Rightarrow \phi(y) = 2/3\, y^3$

Therefore, $F(x,y) = x^2 + x^2 y^2 + 2/3\, y^3$

The solution is given by

$F(x,y) = C$ or

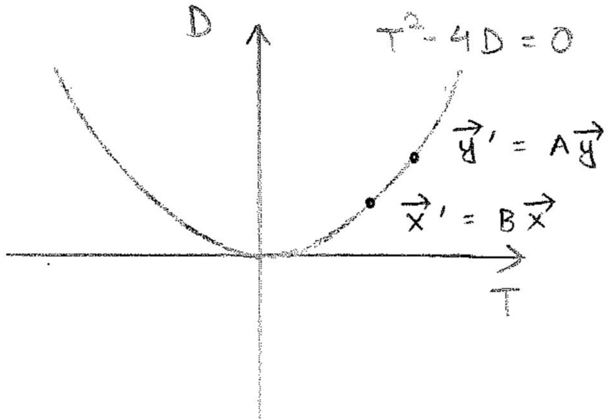
$x^2 + x^2 y^2 + 2/3\, y^3 = C$

Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)



A :

$T = 3 + 3 = 6$

$D = 9$

$T^2 - 4D = 36 - 36 = 0$

B :

$T = 2 + 2 = 4$

$D = 4$

$T^2 - 4D = 0$

Both the systems lie on the parabola $T^2 - 4D = 0$

(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$\lambda = 3 \qquad A - 3I = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$

Let $\vec{v_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (A - 3I)\,\vec{v_1} = \vec{v_2}$

$$\vec{y}(t) = c_1 e^{3t} \begin{pmatrix} -2 \\ 0 \end{pmatrix} + c_2 e^{3t} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right)$$

$$= e^{3t} \begin{pmatrix} -2 \\ 0 \end{pmatrix} (c_1 + t c_2) + e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} c_2$$

$c_1 = 0 \qquad \vec{y}(t) = \cancel{c_3}\; c_2 e^{3t} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right)$

$c_2 = 0 \qquad \vec{y}(t) = c_1 e^{3t} \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

(continued at the end)

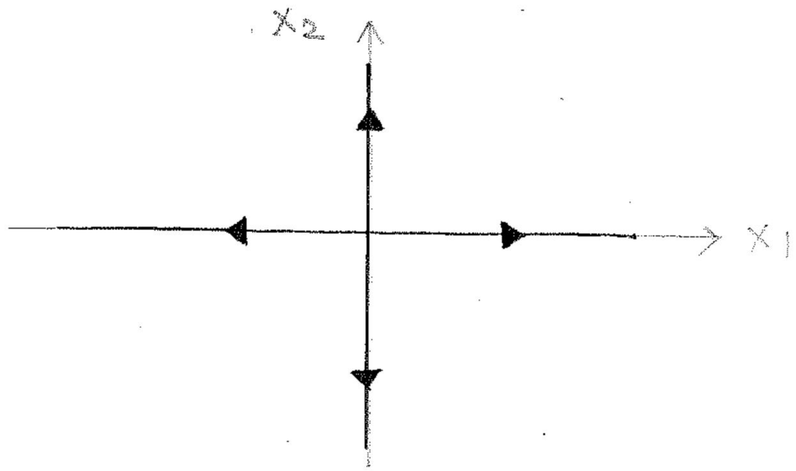
(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$\lambda = 2 \qquad B - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{v}_1, \vec{v}_2 \in \ker(B - 2I)$

$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$c_1 = 0 \qquad \vec{x}(t) = c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$c_2 = 0 \qquad \vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

Let $x^* = y'$ Then $y'' = x'$

Then the planar system of first-order equations is

$y' = x$

$x' = -3t^2x + \cos(t)y - 3e^t$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \ldots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

We can write the n-order equation as a system of first-order equations by making the following substitutions:

$y = y_1$

$y' = y_2$

$y'' = y_3$

$\vdots$

$\cancel{y^{(n)}} \cancel{=} y^{(n-1)} = y_n$ Then $y^{(n)} = y_n'$

Then the system is given by

$y_1' = y_2$

$y_2' = y_3$

$y_3' = y_4$

$\vdots$

$y_n' = F(t, y_1, y_2, \ldots, y_n)$

(9) (2) Continued

Now, we know that $x(t) = 1$ and $x(t) = 2$ are constant solutions. By uniqueness theorem, no other solution can attain the value $x(t_0) = 1$ or $x(t_0) = 2$ for any $t_0 \in (0, \infty)$. Therefore, the solution to the initial value problem ~~is~~ ~~res~~ $x(1) = 1.2$ is restricted to being between 1 and 2 for all $t \in (0, \infty)$. Hence, ~~its~~ it's range is contained in $(1, 2)$

(11) (2) ~~Contained~~ Continued